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# On quotients of Hom-functors (Research on finite groups and their representations, vertex operator algebras, and algebraic combinatorics)

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CITATION:

Tambara, Daisuke. On quotients of Hom-functors (Research on finite groups and their representations, vertex operator algebras, and algebraic combinatorics). 数理解析研究所講究録 2014, 1872: 158-164

ISSUE DATE:

2014-01

URL:

<http://hdl.handle.net/2433/195482>

RIGHT:

## On quotients of Hom-functors

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### 1. Introduction

A hom-functor on a category  $C$  is the functor  $\text{Hom}(-, X)$  for an object  $X$  of  $C$ . We consider the quotient functor  $\text{Hom}(-, X)/G$  by a subgroup  $G$  of  $\text{Aut } X$ . We are interested in replacing hom-functors in the definitions of limit and adjoint by quotients of hom-functors.

### 2. limit

We recall the definition of limit in terms of hom-functor. **Set** denotes the category of sets. For a small category  $C$ ,  $[C^{\text{op}}, \mathbf{Set}]$  denotes the category of contravariant functors  $C \rightarrow \mathbf{Set}$ .  $[C^{\text{op}}, \mathbf{Set}]$  has limits. For instance, the product  $F \times G$  of  $F$  and  $G$  in  $[C^{\text{op}}, \mathbf{Set}]$  is given by

$$(F \times G)(A) = F(A) \times G(A) \quad \text{for } A \in C.$$

And the final object **1** of  $[C^{\text{op}}, \mathbf{Set}]$  is given by

$$\mathbf{1}(A) = \{1\} \quad \text{for } A \in C.$$

For  $X \in C$ , the hom-functor  $h_X$  is defined by

$$h_X(A) = \text{Hom}(A, X).$$

A functor  $F: C^{\text{op}} \rightarrow \mathbf{Set}$  is said to be representable if  $F \cong h_X$  for some  $X$ .

For  $X_1, X_2, Z \in C$  we have

$$Z \text{ is a product of } X_1 \text{ and } X_2 \iff h_Z \cong h_{X_1} \times h_{X_2}.$$

Therefore

$$\begin{aligned} & \text{product of two objects exists in } C \\ \iff & \text{product of two representable functors is representable.} \end{aligned}$$

And similarly

$$\text{a final object exists in } C \iff \mathbf{1} \text{ is representable.}$$

The existence of a limit in  $C$  is thus expressed as the representability of a limit of hom-functors. We first aim to replace representability by familial representability.

### 3. Sum of hom-functors

A functor  $F: C^{\text{op}} \rightarrow \mathbf{Set}$  is said to be *familiarily representable* if

$$F \cong \coprod h_{X_i}$$

for some family  $X_i$  of objects in  $C$  ([Carboni and Johnstone]).

**Theorem 1.** Let  $C$  be a finite category. The following conditions are equivalent to each other.

- (i)  $h_X \times h_Y$  and  $\mathbf{1}$  are familiarily representable ( $\forall X, Y \in C$ ).
- (ii) Finite limits of hom-functors are familiarily representable.
- (iii) Pushouts and coequalizers exist in  $C$ .
- (iv) Finite connected limits exist in  $C$ .

Moreover these conditions imply that all morphisms of  $C$  are epimorphisms.

Remark. “(iii)  $\implies$  (iv)” is generally true.

For the proof of the theorem we may follow the proof of the general representability theorem in [Freyd and Scedrov]. It simplifies owing to our finiteness assumption. We may also use the characterization of familiarily representable functors ([Leinster]).

An interest with such categories comes from an attempt to define general Burnside rings. Suppose that  $C$  satisfies (i) of Theorem 1. For any  $X, Y \in C$  we take isomorphisms

$$h_X \times h_Y \cong \coprod h_{Z_i}$$

and

$$\mathbf{1} \cong \coprod h_{W_j}.$$

Then the free abelian group based on the isomorphism classes of objects of  $C$  becomes a ring by setting

$$\begin{aligned} [X][Y] &= \sum [Z_i], \\ 1 &= \sum [W_j]. \end{aligned}$$

Here  $[X]$  stands for the isomorphism class of an object  $X$ . This ring may be called the Burnside ring of  $C$ .

### 4. The Burnside ring of a finite category

Let  $C$  be a finite category. Assume that  $C$  satisfies the following conditions.

(B1) For every  $X, Y \in C$  there exists a unique family of integers  $c_Z^{XY}$  such that

$$|\text{Hom}(A, X)| |\text{Hom}(A, Y)| = \sum_Z c_Z^{XY} |\text{Hom}(A, Z)| \quad (\forall A \in C).$$

(Here  $|S|$  stands for the cardinality of a set  $S$ .)

(B2) There exists a unique family of integers  $d_Z$  such that

$$1 = \sum_Z d_Z |\operatorname{Hom}(A, Z)| \quad (\forall A \in C).$$

Then the free abelian group based on the isomorphism classes of objects of  $C$  becomes a ring:

$$\begin{aligned} [X][Y] &= \sum_Z c_Z^{XY} [Z], \\ 1 &= \sum_Z d_Z [Z]. \end{aligned}$$

**Theorem.** ([Yoshida]) Assume that a finite category  $C$  satisfies the following conditions.

- (Y1)  $C$  has the unique epi-mono factorization property.
- (Y2)  $C$  has the coequalizer

$$\operatorname{Coeq}(X \xrightarrow[\alpha]{1} X)$$

for any  $\alpha \in \operatorname{Aut} X$ .

Then  $C$  satisfies (B1) and (B2).

The following diagram shows the relationship between Theorem 1 and Yoshida's theorem:

$$\begin{array}{ccc} & [X][Y] = \sum c_Z^{XY} [Z], & \\ \text{pushout, coequalizer exist} \implies & 1 = \sum d_Z [Z], & \\ & c_Z^{XY}, d_Z \in \mathbb{N} & \\ \Downarrow & & \Downarrow \\ \text{epi-mono factorization,} & [X][Y] = \sum c_Z^{XY} [Z], & \\ \operatorname{Coeq}(X \rightrightarrows X) \text{ exist} \implies & 1 = \sum d_Z [Z], & \\ & c_Z^{XY}, d_Z \in \mathbb{Z} & \end{array}$$

A problem will be to characterize categories satisfying (B1) and (B2).

Here are examples of generalized Burnside rings. Let  $G$  be a finite group.

(1) Let  $C$  be the category whose objects are  $G$ -sets  $G/H$  for all subgroups  $H$ , and whose morphisms are  $G$ -maps. Then  $C$  satisfies the condition of Theorem 1. The resulting ring is the ordinary Burnside ring of  $G$ .

(2) Let  $\mathcal{F}$  be a family of subgroups of  $G$  which is closed under conjugation and intersection. Let  $C$  be the category whose objects are  $G$ -sets  $G/H$  for  $H \in \mathcal{F}$ . Then  $C$  satisfies the condition of Theorem 1.

(3) Let  $\mathcal{F}$  be the set of all  $p$ -centric subgroups of  $G$ . Let  $C$  be the category whose objects are  $G$ -sets  $G/H$  for  $H \in \mathcal{F}$ . Then  $C$  satisfies the condition that  $h_X \times h_Y$  are familially representable ([Diaz and Libman], [Oda]). Further examples of  $\mathcal{F}$  are found in [Oda and Sawabe].

(4) For a fusion system  $\mathcal{F}$  a certain category  $\mathcal{O}(\mathcal{F}^c)$  is defined. Then  $C = \mathcal{O}(\mathcal{F}^c)$  satisfies the condition that  $h_X \times h_Y$  are familially representable ([Puig], [Diaz and Libman]).

## 5. Finiteness of connected components of powers of a functor

**FinSet** denotes the category of finite sets. Let  $K$  be a finite category. We say  $G \in [K, \mathbf{FinSet}]$  is connected if  $G$  is nonempty and never expressed as a sum of nonempty objects. Every  $F \in [K, \mathbf{FinSet}]$  is a sum of connected objects, each of which we call a connected component of  $F$ . For  $F \in [K, \mathbf{FinSet}]$  and  $n \geq 0$  we have

$$F^n = F \times \cdots \times F$$

in  $[K, \mathbf{FinSet}]$ .

**Theorem 2.** For  $F \in [K, \mathbf{FinSet}]$ , the following are equivalent.

- (i) Connected components of  $F^n$  for all  $n$  have only finitely many isomorphism classes.
- (ii)  $F(\alpha)$  is injective for every morphism  $\alpha$  of  $K$ .

This theorem relates to Theorem 1 as follows: Let  $F: K \rightarrow \mathbf{FinSet}$  satisfy (ii) of Theorem 2. Let  $C$  be a representative system of isomorphism classes of connected components of  $F^n$  for all  $n$ . Then  $C$  is finite. View  $C$  as a category (a full subcategory of  $[K, \mathbf{FinSet}]$ ). For  $X, Y \in C$ ,  $X \times Y$  is a sum of objects of  $C$  and  $\mathbf{1}$  is a sum of objects of  $C$ . So  $C$  satisfies condition (i) of Theorem 1.

Conversely every finite category satisfying condition (i) of Theorem 1 arises this way.

## 6. Quotient of hom-functor

Let  $C$  be a category. Let  $X$  be an object of  $C$  and  $G$  a subgroup of  $\text{Aut } X$ . We define the functor  $h_X/G: C^{\text{op}} \rightarrow \mathbf{Set}$  by

$$(h_X/G)(A) = \text{Hom}(A, X)/G.$$

Here  $\text{Hom}(A, X)/G$  is the quotient set relative to the natural action of  $G$  on  $\text{Hom}(A, X)$ .

**Theorem 3.** Let  $C$  be a finite category. The following conditions are equivalent to each other.

- (i)  $h_X \times h_Y$  and  $\mathbf{1}$  are isomorphic to sums of quotients of hom-functors ( $\forall X, Y$ ).
- (ii) Finite limits of hom-functors are isomorphic to sums of quotients of hom-functors.
- (iii) Pushouts exist in  $C$ .
- (iv) Finite simply connected limits exist in  $C$ .

These conditions imply that all morphisms of  $C$  are epimorphisms.

Remark. “(iii)  $\implies$  (iv)” is true for a general  $C$  ([Paré]).

## 7. Category with pushouts

We here give an example of a category with pushouts.

Let  $P$  be a partially ordered set. Suppose that a group  $G$  acts on  $P$ :

$$\sigma \in G, x \in P \rightsquigarrow x^\sigma \in P.$$

The category  $PG$  is defined as follows.

(object) Objects of  $PG$  are elements of  $P$ .

(morphism) For  $x, y \in P$

$$\text{Hom}_{PG}(x, y) = \{\sigma \mid \sigma \in G, x \leq y^\sigma\}.$$

(composition) Composition is given by multiplication in  $G$ .

**Proposition.** If  $P$  has pushouts, then so does  $PG$ .

That  $P$  has pushouts means that if  $z \leq x, z \leq y$ , then there exists  $\sup(x, y)$ .

Suppose that for each  $x \in P$  a subgroup  $K_x$  of  $G$  is given. Assume the following conditions hold.

(i)  $\sigma \in K_x \implies x^\sigma = x$

(ii)  $x \leq y \implies K_x \leq K_y$

(iii)  $K_{x^\sigma} = K_x^\sigma$

We then define the category  $D$  as follows.

(object) Objects of  $D$  are elements of  $P$ .

(morphism) For  $x, y \in P$  we set

$$\text{Hom}_D(x, y) = \text{Hom}_{PG}(x, y)/K_y.$$

Here  $K_y$  acts on  $\text{Hom}_{PG}(x, y)$  by multiplication in  $G$ .

(composition) The composition of  $D$  is induced by that of  $PG$ .

**Proposition.** If  $P$  has pushouts, then so does  $D$ .

## 8. Adjoint

We recall the definition of adjoint in terms of hom-functor. Let  $F: B \rightarrow C$  and  $G: C \rightarrow B$  be functors. “ $G$  is a right adjoint of  $F$ ” means

$$\begin{aligned} \text{Hom}_C(F(X), Y) &\cong \text{Hom}_B(X, G(Y)) \\ &\text{(naturally in } X, Y\text{).} \end{aligned}$$

This isomorphism,  $X$  viewed a variable, is written as

$$\begin{aligned} \text{Hom}_C(F(-), Y) &\cong h_{G(Y)} \\ &\text{(naturally in } Y\text{).} \end{aligned}$$

$\text{Hom}_C(F(-), Y) = h_Y \circ F$  denoted by  $F^*(h_Y)$ , this is written as

$$F^*(h_Y) \cong h_{G(Y)}.$$

Thus

$$\begin{aligned} &F \text{ has a right adjoint} \\ \iff &F^*(h_Y) \text{ are representable for all } Y \in C. \end{aligned}$$

We next aim to replace representability in the right-hand side by familial representability.

### 9. Discrete fibration

Recall that a functor  $F: B \rightarrow C$  is called a discrete fibration if the following condition holds.

$$\begin{aligned} &\forall g: F(X) \rightarrow Y' \quad \text{morphism of } C, \\ &\exists! f: X \rightarrow X' \quad \text{morphism of } B, \\ &F(f) = g. \end{aligned}$$

If  $F: B \rightarrow C$  is a discrete fibration, then

$$F^*(h_Y) \cong \coprod_{X \in F^{-1}(Y)} h_X$$

for every  $Y \in C$ .

**Proposition.** Let  $F: B \rightarrow C$  be a functor. The following are equivalent.

- (i)  $F^*(h_Y)$  are familially representable for all  $Y \in C$ .
- (ii) There exists a factorization

$$\begin{array}{ccc} & C' & \\ & \downarrow \pi & \\ F' \nearrow & & \downarrow \pi \\ B & \xrightarrow{F} & C \end{array}$$

such that  $F'$  has a right adjoint and  $\pi$  is a discrete fibration.

### 10. Condition (G)

Here we aim to replace representability in the definition of adjoint by being isomorphic to a sum of quotients of hom-functors.

Let  $F: B \rightarrow C$  be a functor. We introduce the condition (G) for  $F$ . It consists of the following:

(i)

$$\begin{aligned} &g: F(X) \rightarrow Y' \\ \implies &\exists f: X \rightarrow X', F(f) = g. \end{aligned}$$

(ii)

$$\begin{aligned} f_1: X &\rightarrow X'_1, f_2: X \rightarrow X'_2, F(f_1) = F(f_2) \\ \implies \exists u: X'_1 &\rightarrow X'_2, F(u) = 1, f_2 = uf_1. \end{aligned}$$

If condition (G) holds, then  $F^*(h_Y)$  is isomorphic to a sum of quotients of hom-functors for every  $Y \in C$ .

**Theorem 4.** Let  $F: B \rightarrow C$  be a functor. Assume that  $C$  is finite. The following are equivalent.

- (i)  $F^*(h_Y)$  are isomorphic to sums of quotients of hom-functors for all  $Y \in C$ .
- (ii) There exists a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{F'} & C' \\ \nu \downarrow & & \downarrow \pi \\ B & \xrightarrow{F} & C \end{array}$$

such that  $F'$  has a right adjoint,  $\nu$  is full and dense, and  $\pi$  satisfies condition (G).

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